Ricci-flat supertwistor spaces

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ABSTRACT

We show that supertwistor spaces constructed as a Kähler quotient of a hyperkähler cone (HKC) with equal numbers of bosonic and fermionic coordinates are Ricci-flat, and hence, Calabi-Yau. We study deformations of the supertwistor space induced from deformations of the HKC. We also discuss general infinitesimal deformations that preserve Ricci-flatness.

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1   Introduction and Summary

Supertwistor spaces [1, 2] have recently aroused a great deal of interest ([3] and citations thereof). However, a broader discussion of what kinds of supermanifolds can be interpreted as supertwistor spaces does not seem to have been given. Here we consider general super-twistor spaces of supermanifolds whose bosonic part is quaternion Kähler; such twistor spaces admit a canonical Einstein metric, which, as we show, is always Ricci-flat when the number of bosonic dimensions is one less than the number of fermionic dimensions.[1] This is important for the recent applications of supertwistor spaces, which use the technology of topological strings.

Our approach depends on the Swann space [6] or hyperkähler cone (HKC) [7] of a quaternion Kähler supermanifold. It allows us to study certain kinds of new deformations of known supertwistor spaces that could be related to gauge theories other than $N = 4$ super Yang-Mills theory in the conformal phase. Other deformations, involving non-anticommutativity [8], or orbifolding [9, 10] do not appear to fit into our framework.

The plan of the paper is as follows: In section 2, we prove that a supertwistor space constructed as a Kähler quotient of an HKC with equal numbers of bosonic and fermionic dimensions is super-Ricci-flat. In section 3 we show that $\mathbb{CP}^{3|4}$ [3] and the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ [2] arise naturally in our framework–indeed they are the supertwistor spaces on the first two orthogonal Wolf spaces (see, *e.g.,* [11]). In section 4, we discuss general infinitesimal deformations of $\mathbb{CP}^{n-1|n}$, and consider a few examples (this discussion is not based on the HKC). In section 5, we discuss examples of deformations of the HKC. We end with some comments and discussion.

2   HKC's, twistor spaces, and Calabi-Yau's

The Swann space [6] or hyperkähler cone (HKC) [7] is a hyperkähler variety that can be constructed from any quaternion Kähler manifold (QK). The HKC has a homothety and an $SU(2)$ isometry that rotates the hyperkähler structure. The twistor space Z of the QK manifold is the Kähler quotient of the HKC with respect to an arbitrary $U(1)$ subgroup of the $SU(2)$ isometry; this quotient is an S^2 bundle over the QK, where the S^2 parametrizes the choices of the $U(1)$ subgroup used in the quotient.

The twistor space of a quaternion Kähler manifold is always an Einstein manifold. Here we show, using the results of [7], that if the HKC is a super-variety with an equal number of bosonic and fermionic coordinates, then the twistor space is Ricci-flat, and

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[1]For a different approach to studying Ricci-flat supermanifolds, see [4, 5].

hence, Calabi-Yau.

The Kähler potential of an HKC can be written as:

$$\chi = e^{z+\bar{z}+K} \ , \tag{1}$$

where K is the Kähler potential of the twistor space. The resulting metric is

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \chi = \chi \begin{pmatrix} K_{i\bar{j}} + K_i K_{\bar{j}} & K_i \\ & \\ K_{\bar{j}} & 1 \end{pmatrix} \ . \tag{2}$$

As the HKC is hyperkähler, the determinant of this metric is 1 (up to holomorphic diffeomorphisms). However,

$$\det(g_{a\bar{b}}) = \chi^{2n} \det(K_{i\bar{j}}) \ , \tag{3}$$

where $K_{i\bar{j}}$ is the Kähler metric on the twistor space, and $2n$ is the complex dimension of the HKC. Thus, up to holomorphic diffeomorphisms and Kähler transformations,

$$\det(K_{i\bar{j}}) = e^{-2nK} \ . \tag{4}$$

This means that the twistor space is Einstein with cosmological constant $2n$. If the HKC is a *super*-variety with complex dimension $(2n|2m)$, then (3) becomes:

$$\mathrm{sdet}(g_{a\bar{b}}) = \chi^{2(n-m)} \, \mathrm{sdet}(K_{i\bar{j}}) \ , \tag{5}$$

where sdet is the *super*-determinant. The HKC super-variety is Ricci flat, and hence the superdeterminant in (5) can be chosen to be 1 as before. Hence

$$\mathrm{sdet}(K_{i\bar{j}}) = e^{-2(n-m)K} \ , \tag{6}$$

and for $n = m$, the twistor space is Ricci-flat and thus Calabi-Yau. Note that since it is constructed from the HKC by a Kähler quotient, it has one less complex bosonic dimension than the HKC, and thus its complex dimension is $(2n - 1, 2n)$.

Since the techniques for constructing HKC's are rather general, we can thus find many Calabi-Yau twistor supermanifolds. We now turn to some specific examples.

3 Examples

The simplest example of our construction arises for the supertwistor space $\mathbb{CP}^{3|4}$ [3]. In this case, the HKC is simply the flat space $\mathbb{C}^{4|4}$, and the QK manifold is the supersphere $S^{4|4}$ (with 4 real bosonic dimensions and 4 complex fermionic dimensions).

The next example of an HKC that we consider is a hyperkähler quotient [12] of $\mathbb{C}^{8|6}$ by $U(1)$; a nice feature of this example is that the Wick-rotated version admits interesting deformations analogous to those discussed in [13]. Since it is bit less obvious, we'll write this example out. We label the coordinates of $\mathbb{C}^{8|6} \equiv \mathbb{C}_+^{4|3} \times \mathbb{C}_-^{4|3}$ as $z_\pm, \psi_\pm$. The HKC is the variety defined by restriction of the Kähler quotient to the quadric:

$$
\begin{aligned}
\frac{\partial}{\partial V}\hat{\chi} &= 0 , \quad z_+ z_- + \psi_+ \psi_- = 0 , \quad \text{where} \\
\hat{\chi} &= e^V(z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+) + e^{-V}(\bar{z}_- z_- + \bar{\psi}_- \psi_-) .
\end{aligned}
\tag{7}
$$

This gives

$$
\chi = 2\sqrt{(z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+)(\bar{z}_- z_- + \bar{\psi}_- \psi_-)} ,
\tag{8}
$$

where we may choose a gauge, *e.g.*, $z_+^0 = -1$ and solve the constraint by

$$
z_-^0 = \sum_i^3 z_+^i z_-^i + \sum_a^4 \psi_+^a \psi_-^a .
\tag{9}
$$

We find the twistor space by taking a $U(1)$ Kähler quotient of this space; it is actually clearer to start with $\hat{\chi}$; the gauged action is

$$
\widehat{\tilde{\chi}} = e^{\tilde{V}} \left[e^V(z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+) + e^{-V}(\bar{z}_- z_- + \bar{\psi}_- \psi_-) \right] - \tilde{V} ;
\tag{10}
$$

changing variables to $V_\pm = \tilde{V} \pm V$, we have

$$
\widehat{\tilde{\chi}} = e^{V_+}(z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+) + e^{V_-}(\bar{z}_- z_- + \bar{\psi}_- \psi_-) - \frac{(V_+ + V_-)}{2} ,
\tag{11}
$$

which clearly gives the Kähler quotient space $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$; thus the twistor space is the quadric given by $z_+ z_- + \psi_+ \psi_- = 0$ (in homogeneous coordinates) in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$. In this construction, it is clear that the symmetry is $SU(4|3)$. This space has been proposed as an alternate twistor space (*ambitwistor space*) relevant to $N = 4$ super Yang-Mills theory [2, 3, 14]; some recent articles that work with ambitwistors include [15, 16, 17, 18]. It is interesting to study the QK space below this twistor space. It is the supermanifold (with 6 complex fermions!) extension of the QK Wolf-space $\frac{SU(4)}{SU(2) \times U(2)}$.

4 Infinitesimal deformations of $\mathbb{CP}^{n-1|n}$

Since the supertwistor space $\mathbb{CP}^{3|4}$ has played such an important role in our understanding of $N = 4$ super Yang-Mills theory, it is natural to study deformations that preserve the

Ricci flatness of the supermetric. We now describe such general linearized deformations of $\mathbb{CP}^{n-1|n}$ for all n. Consider the ansatz for the Kähler potential

$$K = \ln(1 + z\bar{z} + \psi\bar{\psi}) + \Delta = K_0 + \Delta , \qquad (12)$$

where Δ should be thought of as "small". We calculate the different blocks of the metric:

$$A \equiv K_{z\bar{z}} \qquad (13)$$
$$B \equiv K_{z\bar{\psi}} \qquad (14)$$
$$C \equiv K_{\psi\bar{z}} \qquad (15)$$
$$D \equiv K_{\psi\bar{\psi}} . \qquad (16)$$

The Ricci tensor is given by the complex Hessian of the logarithm of the superdeterminant

$$\mathrm{sdet}(\partial\bar{\partial}K) = \frac{\det(A - BD^{-1}C)}{\det D} , \qquad (17)$$

so the condition that the manifold is Ricci-flat implies (up to holomorphic diffeomorphisms) that the superdeterminant is a constant. We first check this for $\Delta = 0$:

$$A_0 = e^{-K_0}\left(1 - \bar{z} \otimes z e^{-K_0}\right) \qquad (18)$$
$$B_0 = -e^{-2K_0}\bar{z} \otimes \psi \qquad (19)$$
$$C_0 = -e^{-2K_0}\bar{\psi} \otimes z \qquad (20)$$
$$D_0 = e^{-K_0}\left(1 - \bar{\psi} \otimes \psi e^{-K_0}\right) . \qquad (21)$$

From these expressions we find

$$D_0^{-1} = e^{K_0}\left(1 + \frac{\bar{\psi} \otimes \psi}{1 + z\bar{z}}\right) \qquad (22)$$

$$A_0 - B_0 D_0^{-1} C_0 = e^{-K_0}\left(1 - \frac{\bar{z} \otimes z}{1 + z\bar{z}}\right) \qquad (23)$$

$$\left(A_0 - B_0 D_0^{-1} C_0\right)^{-1} = e^{K_0}\left(1 + \bar{z} \otimes z\right) , \qquad (24)$$

which immediately leads to the bosonic determinant

$$\det\left(A_0 - B_0 D_0^{-1} C_0\right) = e^{(1-n)K_0}(1 + z\bar{z})^{-1} . \qquad (25)$$

The fermionic determinant is slightly more complicated

$$\det D_0 = e^{-nK_0}e^{\mathrm{Tr}\ln(1 - \bar{\psi}\otimes\psi e^{-K_0})} = e^{(1-n)K_0}(1 + z\bar{z})^{-1} . \qquad (26)$$

so we see that $\mathrm{sdet}(\partial\bar{\partial}K) = 1$.

4

We now repeat the calculation for a small but nonzero Δ. Then the superdeterminant should be calculated with $A = A_0 + A_1$, $B = B_0 + B_1$, ... where A_1, B_1, C_1, D_1 come entirely from Δ. The condition that the superdeterminant is one implies:

$$\det D = \det \left(A - BD^{-1}C \right) , \tag{27}$$

which, to lowest order in the small Δ becomes

$$\begin{aligned} \mathrm{Tr} D_0^{-1} D_1 &= \mathrm{Tr} \left[\left(A_0 - B_0 D_0^{-1} C_0 \right)^{-1} \right. \\ &\quad \left. \times \left(A_1 - B_1 D_0^{-1} C_0 - B_0 D_0^{-1} C_1 + B_0 D_0^{-1} D_1 D_0^{-1} C_0 \right) \right] . \end{aligned} \tag{28}$$

This can be simplified by inserting the explicit expressions for the unperturbed matrices:

$$\mathrm{Tr} D_1 = \mathrm{Tr} A_1 + z A_1 \bar{z} + z B_1 \bar{\psi} + \psi C_1 \bar{z} + \psi D_1 \bar{\psi} . \tag{29}$$

An example of a solution to this equation is found in the next section. Since the full bosonic $SU(n)$ symmetry is nonlinearly realized, all nontrivial deformations appear to necessarily break it, and it is unclear what if any applications for Yang-Mills theories such deformations may have.[2] An intriguing possibility is that some such deformation could describe the nonconformal phase of $N = 4$ super Yang-Mills theory.[3]

5 Examples of deformations of the HKC

HKC's that are constructed as hyperkähler quotients can be deformed by deforming the quotient (see, *e.g.*, [13]). Both the examples described in section 3 can be deformed; there are many deformations of the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ (ambitwistor space), and we begin a few examples.

5.1 Some deformations of ambitwistor space

The simplest kind of deformations involve modifying the $U(1)$ charges of the fermions when we perform the hyperkähler quotient, that is (7). Instead of the diagonal $U(1)$, we

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[2]The most general ansatz for an infinitesimal perturbation that *does* preserve $SU(n)$ symmetry has the form

$$\Delta = \sum_{i=1}^{n} \sum_{j_1 \ldots j_i \bar{k}_1 \ldots \bar{k}_i = 1}^{n} a^i_{j_1 \ldots j_i \bar{k}_1 \ldots \bar{k}_i} \frac{\psi^{j_1} \ldots \psi^{j_i} \bar{\psi}^{k_1} \ldots \bar{\psi}^{k_i}}{(1 + z \cdot \bar{z} + \psi \cdot \bar{\psi})^i} ,$$

where the a^i are hermitian arrays $a^i_{j_1 \ldots j_i \bar{k}_1 \ldots \bar{k}_i} = \bar{a}^i_{\bar{k}_1 \ldots \bar{k}_i j_1 \ldots j_i}$; we have checked that for $n = 3$, eq. (29) implies $\Delta = \sum_{ij} (a_{ij} \psi^i \bar{\psi}^j)/(1 + z \cdot \bar{z} + \psi \cdot \bar{\psi})$, which can be absorbed by a holomorphic change of coordinates $\psi \to \psi + \frac{1}{2} a_{ji} \psi^j$.

[3]We thank R. Wimmer for this suggestion.

5

may use any subgroup of the $U(3)$ that acts on the fermion hypermultiplets. For example, we may choose a $U(1)$ that acts only on one or two pairs of the fermions; if we label the fermionic coordinates as $\psi_\pm, \psi_0$ according to their $U(1)$ charge, (7) becomes

$$
\begin{aligned}
\frac{\partial}{\partial V}\hat{\chi} &= 0 , \quad z_+ z_- + \psi_+ \psi_- = 0 , \quad \text{where} \\
\hat{\chi} &= e^V (z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+) + e^{-V}(\bar{z}_- z_- + \bar{\psi}_- \psi_-) + \psi_0 \bar{\psi}_0 .
\end{aligned}
\tag{30}
$$

This gives

$$
\chi = 2\sqrt{(z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+)(\bar{z}_- z_- + \bar{\psi}_- \psi_-)} + \psi_0 \bar{\psi}_0 .
\tag{31}
$$

We find the twistor space by taking a $U(1)$ Kähler quotient of this space; it is actually clearer to start with $\hat{\chi}$; the gauged action is

$$
\widehat{\widetilde{\chi}} = e^{\tilde{V}} \left[e^V (z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+) + e^{-V}(\bar{z}_- z_- + \bar{\psi}_- \psi_-) + \psi_0 \bar{\psi}_0 \right] - \tilde{V} .
\tag{32}
$$

In homogeneous coordinates, this gives the quadric $z_+ z_- + \psi_+ \psi_- = 0$ in Kähler super-manifold with Kähler potential

$$
K = \frac{1}{2}\ln(M_+) + \frac{1}{2}\ln(M_-) + \ln\left(1 + \frac{\psi_0 \bar{\psi}_0}{2\sqrt{M_+ M_-}}\right) .
\tag{33}
$$

where

$$
M_+ = (z_+ \bar{z}_+ + \psi_+ \bar{\psi}_+) , \quad M_- = (\bar{z}_- z_- + \bar{\psi}_- \psi_-) .
\tag{34}
$$

This deformation of the ambitwistor space has the symmetry $SU(4|n) \times SU(6-2n)$, where n is the number of charged doublets $\psi_\pm$.

Other choices of the $U(1)$ action on the fermions give other deformations with different residual symmetries.

If we change the metric of the initial flat space whose $U(1)$ quotient gives the HKC, e.g., so that the global symmetry group acting on the fermions becomes $U(2,1)$, then new deformations analogous to those discussed in [13] become available.

Another kind of deformation arises from the observation that the underlying bosonic quaternion Kähler manifold, the unitary Wolf space $SU(4)/[SU(2) \times SU(2) \times U(1)]$ is the same as the orthogonal Wolf space $SO(6)/[SO(3) \times SO(3) \times SO(2)]$, and hence, as explicitly shown in [11], the corresponding HKC's are equivalent. These deformations are rather complicated, and lead to a variety of residual symmetry groups.

5.2 Deformations of $\mathbb{CP}^{3|4}$

The chiral twistor space $\mathbb{CP}^{3|4}$ appears to be more useful than ambitwistor space, *e.g.*, [3, 19, 20, 21]. However, in contrast to the deformations of ambitwistor space, it is not obvious that any of the deformations of $\mathbb{CP}^{3|4}$ preserve the bosonic conformal symmetry, limiting their possible applications.

The deformations that we consider are similar to the last kind mentioned for the ambitwistor space: namely, deformations based on the identification of S^4 as both the first symplectic Wolf space $Sp(2)/[Sp(1) \times Sp(1)]$ as well as the first orthogonal Wolf space $SO(5)/[SO(3) \times SO(3)]$. This identification implies that the corresponding HKC's are equivalent, and this was indeed explicitly shown in [11]. Thus $\mathbb{C}^4 \equiv \mathbb{C}^{10}///SU(2)$, the hyperkähler quotient of $\mathbb{C}^{10}$ by $SU(2)$, and, as also shown in [11], it is natural to write this as $\mathbb{C}^{20}///[SU(2) \times U(1)^5]$. This suggests a variety of deformations that we now describe.

Two deformations that we consider arise by coupling the $U(1)$'s and the $SU(2)$ to the fermions as follows (we use the language of $N = 1$ superspace σ-model Lagrangians)

$$L_f = \int d^4\theta \, \psi e^{\sigma V} \bar{\psi} e^{q \sum_i V_i} + \tilde{\psi} e^{-\sigma V} \tilde{\bar{\psi}} e^{-q \sum_i V_i} + \left[\int d^2\theta \, \sigma \psi \phi \tilde{\psi} + q \psi \tilde{\psi} \sum_i \phi_i + c.c. \right] ,$$

(35)

where $\sigma = 0, 1$ and q is arbitrary.

5.3 The $U(1)$ deformation

To begin with we set $\sigma = 0$ and study the $U(1)$ deformation. If we integrate out the M'th $U(1)$ vector multiplet we get the following two equations

$$
\begin{aligned}
0 &= z^M \tilde{z}^M + q \psi \tilde{\psi} \\
0 &= z^M e^V \bar{z}^M e^{V_M} - \bar{\tilde{z}}^M e^{-V} \tilde{z}^M e^{-V_M} + q \psi \bar{\psi} e^{q \sum_M V_M} - q \bar{\tilde{\psi}} \tilde{\psi} e^{-q \sum_M V_M} .
\end{aligned}
$$

(36)

We introducing the notation

$$M_M = z^M e^V \bar{z}^M \quad , \qquad \tilde{M}_M = \bar{\tilde{z}}^M e^{-V} \tilde{z}^M .$$

(37)

Though the deformation makes sense, to simplify our calculations, we work only to the lowest nontrivial order in q throughout this subsection. This means that we can compare directly to the results of section 4. Then the second equation of motion simplifies to

$$M_M e^{V_M} - \tilde{M}_M e^{-V_M} + q \left(\psi \bar{\psi} - \bar{\tilde{\psi}} \tilde{\psi} \right) + \mathcal{O}(q^2) = 0 .$$

(38)

Solving for V_M we get

$$V_M = \ln \sqrt{\frac{\tilde{M}_M}{M_M}} - \frac{q(\psi\bar{\psi} - \bar{\tilde{\psi}}\tilde{\psi})}{2\sqrt{M_M\tilde{M}_M}} + \mathcal{O}(q^2) \,. \tag{39}$$

Plugging this back into the action we get

$$\int d^4\theta \left(\sum_{M=1}^{5} 2\sqrt{M_M\tilde{M}_M} + \psi\bar{\psi} + \bar{\tilde{\psi}}\tilde{\psi} + \frac{q}{2}\left(\psi\bar{\psi} - \bar{\tilde{\psi}}\tilde{\psi}\right) \ln \frac{\Pi_{j=N}^{5}\tilde{M}_N}{\Pi_{K=1}^{5}M_K} \right.$$
$$\left. + \int d^2\theta \sum_M z^M \phi \tilde{z}^M + \mathcal{O}(q^2) \,. \right) \tag{40}$$

It is not possible to solve the first equation in (36) in an $SU(2)$ covariant way. Rather we set

$$\tilde{z}^M = i\sigma_2 z^M - q\psi\tilde{\psi}\frac{\sigma_1 z^M}{(z^M \sigma_1 z^M)} + \mathcal{O}(q^2) \,. \tag{41}$$

Inserting this and keeping only terms to lowest nontrivial order in q we get

$$\int d^4\theta \left(\sum_M \left(2z^M e^V \bar{z}^M + q\psi\tilde{\psi}\frac{z^M \sigma_3 e^V \bar{z}^M}{z^M \sigma_1 z^M} + q\bar{\tilde{\psi}}\tilde{\psi}\frac{z^M e^V \sigma_3 \bar{z}^M}{\bar{z}^M \sigma_1 \bar{z}^M} \right) + \psi\bar{\psi} + \bar{\tilde{\psi}}\tilde{\psi} \right)$$
$$+ \int d^2\theta \sum_M \left(z^M \phi i\sigma_2 z^M - q\frac{z^M \phi \sigma_1 z^M}{z^M \sigma_1 z^M}\psi\tilde{\psi} \right) + \mathcal{O}(q^2) + c.c. \tag{42}$$

It is now straightforward to proceed to integrate out the $SU(2)$ gauge field leading to the holomorphic constraints

$$\begin{pmatrix} \sum_M z_+^M z_+^M & \sum_M z_+^M z_-^M \\ \sum_M z_-^M z_+^M & \sum_M z_-^M z_-^M \end{pmatrix} = \frac{q}{2}\psi\tilde{\psi}\begin{pmatrix} -\sum_M \frac{z_+^M}{z_-^M} & 0 \\ 0 & \sum_M \frac{z_-^M}{z_+^M} \end{pmatrix}, \tag{43}$$

together with the action

$$4\int d^4\theta \sqrt{\det \sum_M \left(\bar{z}^M \otimes z^M + \frac{q}{2}\frac{\psi\tilde{\psi}}{z^M \sigma_1 z^M}\bar{z}^M \otimes z^M \sigma_3 + \frac{q}{2}\frac{\bar{\tilde{\psi}}\tilde{\psi}}{\bar{z}^M \sigma_1 \bar{z}^M}\sigma_3 \bar{z}^M \otimes z^M \right)}$$
$$+ \psi\bar{\psi} + \bar{\tilde{\psi}}\tilde{\psi} \,. \tag{44}$$

The holomorphic constraint can be solved as

$$z_\pm^M = w_\pm^M \mp e_\pm^M \frac{q}{4}\psi\tilde{\psi}\sum_N \frac{w_\pm^N}{w_\mp^N} + \mathcal{O}(q^2) \,, \tag{45}$$

where we have used the solutions $w_\pm$ of the $q = 0$ constraints in terms of four complex coordinates u_i introduced in [11]

$$w_+^M = \left(-2u_3, -i\frac{u_3^2}{u_1} - iu_1, -\frac{u_3^2}{u_1} + u_1, \frac{u_3 u_4}{u_1} + u_2, -i\frac{u_3 u_4}{u_1} - iu_2\right)$$

$$w_-^M = \left(2u_4, i\frac{u_3 u_4}{u_1} - iu_2, \frac{u_3 u_4}{u_1} - u_2, -\frac{u_4^2}{u_1} + u_1, i\frac{u_4^2}{u_1} + iu_1\right) \qquad (46)$$

and the two vectors e_+ and e_- are defined such that

$$e_+ \cdot w_+ = e_- \cdot w_- = 1$$

$$e_+ \cdot e_+ = e_- \cdot e_- = e_+ \cdot w_- = e_- \cdot w_+ = e_+ \cdot e_- = 0 \qquad (47)$$

with the explicit solutions

$$e_+ = \left(0, \frac{i}{2u_1}, \frac{1}{2u_1}, 0, 0\right)$$

$$e_- = \left(0, 0, 0, \frac{1}{2u_1}, -\frac{i}{2u_1}\right) \qquad (48)$$

To show that this indeed gives a HKC one has to rewrite this expression in the form

$$e^{u + \bar{u} + K(z, \bar{z}, \psi, \bar{\psi})} \qquad (49)$$

The variable which we factor out and which will play the role of u is u_1. The variables on which K depend are then the original variables rescaled by u_1. One can show that K will be of the form (12) with a Δ looking like

$$\Delta = \frac{f(u, \bar{u})\psi\tilde{\psi} + \bar{f}(u, \bar{u})\bar{\tilde{\psi}}\bar{\psi}}{1 + |u|^2 + |\psi|^2 + |\tilde{\psi}|^2} \qquad (50)$$

where is $f(u, \bar{u})$ is a particular function of the bosonic variables; (29) implies that f has to satisfy the differential equation

$$\frac{\partial^2}{\partial u^i \partial \bar{u}^i} f + u^i \bar{u}^k \frac{\partial^2}{\partial u^i \partial \bar{u}^k} f - u^i \frac{\partial}{\partial u^i} f + \bar{u}^i \frac{\partial}{\partial \bar{u}^i} f - f = 0 . \qquad (51)$$

A straightforward but lengthy calculation confirms that the particular f one gets from the $U(1)$ deformation does indeed satisfy (51).[4]

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[4]Though (50) resembles the $SU(4)$ invariant ansatz noted in section 4, there is no choice of $f(u, \bar{u})$ which is consistent with the symmetry.

5.4 $SU(2)$ **deformations**

In this section we put $q = 0$. Then the holomophic constraints from integrating out the M'th $U(1)$ is

$$z_{Ma}\tilde{z}_M^a = 0 \tag{52}$$

This is solved by putting $\tilde{z}_M^a = (i\sigma_2)^{ab} z_{bM}$. To get a more unified notation, let us also introduce $\psi_a = \frac{1}{\sqrt{2}}(\psi_{1a} - i\psi_{2a})$, $\tilde{\psi}^a = \frac{1}{\sqrt{2}}(i\sigma_2)^{ab}(\psi_{1b} + i\psi_{2b})$. Thus, integrating out the $U(1)$'s and using this new notation, the bosonic piece of the action becomes

$$L_b = \sum_M \left(\int d^4\theta z_M e^V \bar{z}_M + \left[\int d^2\theta z_M \phi i\sigma_2 z_M + c.c. \right] \right) \tag{53}$$

and the fermionic piece becomes

$$L_f = i \int d^4\theta \psi_1 e^V \bar{\psi}_2 - \psi_2 e^V \bar{\psi}_1 + \left[\frac{i}{2} \int d^2\theta \psi_1 \phi i\sigma_2 \psi_2 - \psi_2 \phi i\sigma_2 \psi_1 + c.c. \right] \tag{54}$$

Integrating out the $SU(2)$ gauge field, and letting the $SU(2)$ indices run over $+$ and $-$, the holomorphic constraints can be written as

$$
\begin{aligned}
z_+ \cdot z_+ + 2\psi_+^1\psi_+^2 &= \\
z_+ \cdot z_- + \psi_+^1\psi_-^2 + \psi_-^1\psi_+^2 &= 0 \\
z_- \cdot z_- + 2\psi_-^1\psi_-^2 &=
\end{aligned}
\tag{55}
$$

Again using the variables (46) and (48) we were able to write down an explicit solution to the holomorphic constraints

$$
\begin{aligned}
z_+^M &= w_+^M - e_+^M \psi_+^1\psi_+^2 - e_-^M \psi_-^1\psi_+^2 \\
z_-^M &= w_-^M - e_-^M \psi_-^1\psi_-^2 - e_+^M \psi_+^1\psi_-^2
\end{aligned}
\tag{56}
$$

Now we turn to integrating out V. This can be done with the result

$$\sqrt{\det\left(\bar{z}\cdot z - \bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1\right)} \tag{57}$$

where the determinant is taken over the $+$ and $-$ indices. In this expression one furthermore has to use the expressions of z in terms of the u variables given above. By introducing supercoordinates

$$\phi_\pm^A = \begin{pmatrix} z_\pm^M \\ \psi_\pm^i \end{pmatrix} \tag{58}$$

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and the supermetric

$$g_{AB} = \begin{pmatrix} \delta_{MN} & 0 \\ 0 & \epsilon_{ik} \end{pmatrix} \tag{59}$$

one may rewrite the $SU(2)$ matrix over which we are taking the determinant in a manifestly $OSp(5|2)$ covariant way

$$\begin{pmatrix} \phi_+^A g_{AB} \phi_+^B & \phi_+^A g_{AB} \phi_-^B \\ \phi_-^A g_{AB} \phi_+^B & \phi_-^A g_{AB} \phi_-^B \end{pmatrix} \tag{60}$$

In the case without fermions it was shown in [11] that the manifest $O(5)$ symmetry is enhanced to an $SU(4)$ symmetry. We do not know if this occurs in the presence of the fermions; one might hope that the symmetry is enhanced to the supergroup $SU(4|2)$, which is the superconformal group of of $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ Yang-Mills theory with $2N_c$ hypermultiplets in the fundamental representation. We have not found evidence one way or the other.

To show that this indeed gives a HKC we again have to rewrite this expression in the form

$$e^{u + \bar{u} + K(z, \bar{z}, \psi, \bar{\psi})} \tag{61}$$

Again the variable that we choose to factor out is u_1 and the remaining variables of the Kähler potential are the original variables rescaled by u_1. We thus find a compact form for the Kähler potential of the twistor space. It is however not of the form (12) since the perturbation in no way can be thought of as being "small".

6 Conclusions

We have shown that supertwistor spaces constructed from hyperkähler cones with equal numbers of bosonic and fermionic coordinates are super-Ricci-flat. We have used this result to discuss deformations of supertwistor spaces. All the deformations of the chiral supertwistor space $\mathbb{CP}^{3|4}$ that we found appear to break conformal invariance (though one case we discussed is unclear), whereas it is simple to deform ambitwistor space (the quadric in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$) in ways that preserve a variety of superconformal symmetries. It would be interesting to study if any of these deformations arise in superconformal Yang-Mills theories.

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